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UNIFIED MICRO/MACRO-ECONOMIC THEORY LINKING PRODUCTIVITY TO RISK AND COMPLETING INPUT/OUTPUT SUBSTITUTION

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ABSTRACT

In his pivotal contributions during the marginal revolution, Leon Walras along with W.S. Jevons assigned subjective utility directly to commodities (goods and services) as, in effect, a simplifying assumption—an assumption destined to become the *keystone* of neoclassical economics. But this "keystone" assumption undermined *consumption-duration* as an important—in fact, essential—variable in economic behavior, by excluding it from the utility (satisfaction) function to avoid double-counting. In the present contribution we extend orthodox neoclassical mathematical economics by accounting for generalized consumption-duration while retaining the inviolable commodity-utility postulate—where the new constraint unites Gossenian and neoclassical mathematical economics after a 135 year separation. This step, in turn, permits a complete and explicit formulation of the basic durations (work, consumption, and leisure/rest) within the short-term (e.g., single-day) interval. The resulting formulation is then extended over the long-run to the intertemporal horizon for a postulated *periodic-equilibrium* based on *nested-characteristic-times*—where single-day economic function is unaffected by long-run (multi-day) utility discounting, which in turn is near 100% complete before the entire economic system significantly changes. Within this new general mathematical theory it is shown that neither orthodox neoclassical theory nor (the deeper, more fundamental) Gossenian theory can correctly model the fixed commodity-amount special case—relevant, for example, to the monthly "food-stamp" allocation (now assisting ~15% of the US population.) As an application, a long-run relationship between labor/capital marginal productivities and long-run utility discounting is developed. Two additional applications conclude the paper: (1) Walras' input/output substitution relations are completed using the present short-term system; and (2) forward substitution relations are formulated using the newly-derived long-run (intertemporal) system. As part of (2), the natural interest rate is shown equal to the long-run discount rate for the economic system in equilibrium. …As a caveat to the theorist/analyst, retention of the neoclassical commodity-utility (keystone) assumption—instead of the exclusive identification of instant-utility (feeling-state) with human activity—will require analyst-specified consumption-duration curves for determinate solutions.

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I. INTRODUCTION

J. M. Keynes in his 1936 book "*The General Theory of Employment, Interest and Money*" observed that people are significantly irrational in the "business of life (A. Marshall)":

> "Even apart from the instability due to speculation, there is the instability due to … spontaneous optimism rather than mathematical expectations... Most, probably, of our decisions to do something positive … can only be taken as the result of animal spirits – a spontaneous urge to action rather than inaction – and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities."

From this one might conclude that deeper theoretical insight into economic behavior is of no value—that we may do scarcely more than simply "muddle through".

But there are considerations which suggest an enlightened approach may be feasible. People are in this regard at least partly rational, and we frequently account for outcome-probability—both objectively and subjectively [e.g., instinctively or intuitively]—in our expectational thinking. Moreover, there is indeed a systematic cause-to-effect nature of *productive economic function* on which our discernable rationality can act. Economic theory may accordingly provide some beneficial traction. It is with this positive attitude that the present work on unifying and extending economic theory is offered.

Time is the major concern in the present theoretical study, as it is in real-life economic behavior. In advancing the modeling of time in economics a review of the origins of basic theory as well as periods of major progress is appropriate. The contributions of Leon Walras in his profound and pivotal 1874-77 work are prominent in this regard. Walras, probably more than anyone, established the course of (neoclassical) economic theory in the twentieth century by the particular way he introduced subjective-utility (*satisfaction*, or *time-integrated feeling-state*) into his formulation of the competitive economic system in equilibrium, including input/output substitution: he (along with W.S. Jevons, 1871) identified utility directly with commodities, as in the quantity of utility (i.e., agent <u>cardinal</u> *satisfaction*[1]) per unit commodity at the margin entering the individual's utility calculus.

The commodity-utility assumption soon became the inviolable *keystone* of neoclassical mathematical economics—thereby superseding the more fundamental *human consumptive-activity* as the (epistemologically) proper basis for consumption utility in mathematical economics. An important consequence is the almost total absence of human consumptive-activity—called *consumption-duration*

[1] It may be helpful to the reader to substitute "satisfaction" for "utility" whenever the latter is encountered in the text (or anywhere else in the literature). The term "utility" developed or evolved from meaning one thing—*useful*—to meaning another thing—*time-integrated feeling-state: satisfaction*—in economics during the nineteenth century. So while mathematical economists should be comfortable with thinking of "utility" as satisfaction, this (from personal experience) is not natural or normal for the scholar seeking to understand economic theory.

herein—from economic theory and analysis. Accounting for consumption-duration as an explicit variable in basic and applied neoclassical theory is a primary task in the present work.

Theory Comparison. It is beneficial to compare the present theory of this paper with similar theory in the literature, and a comparison with Ian Steedman's *Consumption Takes Time* (2001) is appropriate for this purpose. We may first note that Steedman believes standard theoretical economics should recognize consumption-duration and its (instant) utility, and I of course agree—notwithstanding my retention herein of the neoclassical commodity-utility assumption (in order to unify Gossenian theory[2] with Walrasian/neoclassical theory).

We may also note that Steedman focused on the "economic theory of consumption" and only secondarily on "…the positive theory of consumption and labor supply that must be reworked." No attempt was made at a *holistic* formulation (which is characteristic of standard/mainstream economics): reduced and simplified models were directly employed to help explanations.[3] This is in contrast to the present approach where the essential variables are drawn together in an overarching theory (as is standard practice in physics, subject to empirical falsification).

It is appropriate to note here that the present Gossenian approach is profoundly different from the standard approach in mathematical economics, and it is important to address the differences—at the risk of some repetition:

> ***Gossenian Approach is Deeper***. *In standard economics it is the continuing belief, for many decades, that different theoretical foundations are required for different socioeconomic systems—as, for example, regarding socialism versus capitalism. Furthermore, as any given socioeconomic system evolves its basic equations must correspondingly evolve.*
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> *I strongly disagree with this philosophy—as the likely consequence of my formal education in natural science (physics and engineering) combined with a long-term, broad and deep study of economic behavior. Theoretical economics, in this regard, is applied*

[2] "Gossenian theory" refers to economic theory developed (over the past 160 years) starting from the fundamental principles by Hermann Gossen in his 1854 book. In this regard, the *Gossen equation* is a mathematical statement of human behavior (*mathematical behavior*), and only acquires relevance to economic behavior through specified assumptions and constraints (as, for example, through definition of labor-activity, consumption-activity, and leisure/rest, and requiring the sum equal 24 hours). Because of the more fundamental (psychological) basis of Gossenian economic theory, assumptions and approximations attending economic modeling are subject to more rigorous criticism and validation.

[3] So-called Dynamic Stochastic General Equilibrium (DSGE) suggests the holistic character. But: (1) no economy is ever in equilibrium; and (2) to write and develop DSGE with insufficient regard for fundamental correctness is to invite misfortune. (Were the development of aerodynamic theory similarly careless, airplanes would stall and crash, without "expert" agreement on the cause!)

> *psychology in exactly the same sense that theoretical meteorology is applied physics. I accordingly followed the scientific method of physics in completing the Gossen equation (completed in 1993—see (2006/7a); empirical support was received the following year from Damasio (1994): "Decartes' error: emotion, reason, and the human brain.")*
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> *Starting from the Gossen equation any economic system can be modeled—whether capitalist or socialist at one extreme to the hunter-gatherer tribe or solitary individual (Crusonian) at the other extreme. Models are accordingly defined using simplifications and constraints on the Gossen equation, qualifications that must be justified to the satisfaction of the scientific community, as is the case in applied physics. Becker in his 1965 article ("A theory of the allocation of time", addressed herein) would then formally justify ignoring consumption-duration and leisure/rest-duration in his model, as well as ignoring the utility attending work-duration; Ramsey (1928) would justify ignoring (variable) consumption-duration and leisure/rest-duration; Walras would justify ignoring work utility (which Jevons recognized); and Jevons would justify ignoring leisure/rest (which Walras recognized). Based on this method rigorously applied, economics would advance as practitioners critique each other's work from a common foundation, thereby converging on a coherent, internally consistent theory.*
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> ***Gossenian Foundation is More Realistic**. Closely related to the deeper foundation is the greater realism of Gossenian theory. In this regard, traditional neoclassical theory, from which modern theory has evolved, falsely assumes at its foundation (perfect) rationality in agent decisions. Following the successful approach of the natural sciences, and that of physics in particular, the Gossenian approach rejects perfect-rationality as a basic assumption and begins on the comprehensive foundation that accommodates both human rationality and irrationality. Then, adopting the same methodology of applied physics and engineering, models are defined using simplifying assumptions and postulated constraints, with justifications, as appropriate to the objective.[4] (Further addressed on page 12-13.)*

In advancing basic (utility) theory Steedman emphasized identifying utility with consumption-rate in addition to identifying with Gossenian consumption-duration[5]. Regarding my own contributions to basic theory, the agent's *expectational* uncertainty (leading to *derived* investment risk) has been

[4] In addition to examples in this paper, more extensive illustrations are provided in Chamberlain (2003/4).
[5] Consumption-rate and consumption-duration are both inherent in the Gossen equation, with both varying at the margin in plan optimization. Steedman's important contribution here is to emphasize/stress this vital aspect of Gossenian mathematical economics.

formulated by way of an unbounded number of "perfect foresight" scenarios/worldlines,[6] each with an expectational occurrence-probability. Taken together, the entire ensemble covers all the (expectedly) possible action-scenarios and outcomes that the individual expects may happen in the candidate or operational plan (seriously considered; Shackle, 1958). Additionally, the dimension [TIME^{-1}] has been assigned to the (previously dimensionless) intertemporal (long-run) utility discount coefficient, thereby having decisions depend on real-time feeling-state rather than expected discounted-utility (optional to the analyst).[7] And, as earlier indicated, the holistic approach is promoted—with a prime objective of advancing the integration of micro-theory and macro-theory for both explaining systemic economic behavior and guiding financial/economic legislation.

Applied theory comprises a further step. Here Steedman's methodology—again, working immediately with simplified models—can offer little in explaining macroeconomic function and dysfunction. However the present *holistic* theory of agent economic-behavior within the macro-economy opens the door to mathematical hypotheses and approximations which can lead to new insights and more effective institutions and policies. *Nested-characteristic-times* is one such hypothesis (and approximation)…having a negligible effect on utility-discounting across the short-term interval (e.g., the single day—used herein) but a vital effect on utility-discounting—and therefore economic function generally—across the long-run interval to the discount-horizon.[8] This (simplifying) hypothesis has allowed a near-equilibrium analysis of market-caused inequality, leading to suggested policy measures for arresting the instability (see 2003/4).

Leisure/Rest is Absent from Neoclassical Theory. In addition to consumption-duration there should be some discussion of leisure/rest in this introduction. This is a subject largely ignored in neoclassical mathematical economics…but not at its beginning: Walras recognized leisure/rest and its utility, albeit indirectly, in his formulation of the competitive economy in equilibrium. Since then the important variable has been overlooked (again, in neoclassical mathematics), and for good reason: leisure/rest-duration has been overlooked because insertion in the analyst's model would immediately force a profoundly unrealistic condition. What is this condition? Because consumption activity (involving

[6] "Perfect foresight" in this paper doesn't mean the individual correctly foresees the future for all time, as is the case in standard/traditional economics. It means the individual's expectational plan (candidate or operational)—accounting for uncertainty/risk—is exactly foreseen for all *intertemporal* time. This expectational foresight will agree with, or overlap, actual future experience up to some unexpected/surprising event whereupon a new plan is immediately initiated (and immediately subject to refinement/revision). …How does this differ from orthodox neoclassical theory? Traditional N/C theory starts (as indicated) from the false assumption that the individual exactly foresees all future/actual experience.
[7] The reader may refer to Chamberlain (2003c) for discussion of these aspects of basic theory.
[8] The language "short-term" and "long-run" is adopted herein in place of "intratemporal" and "intertemporal" for clarity. First, the more concise language is easier to read. And second, the language is more meaningful—where "short-term" refers to the short interval (e.g., single day) across which discounting is negligible and "long-run" refers to the time-extended interval where discounting is important and ultimately complete.

both consumption-rate and consumption-duration) is missing as an explicit variable in orthodox neoclassical economics, were leisure/rest-duration to be mathematically modeled with its positive utility, leisure/rest would simply complement work-duration to fill the 24-hour day: the agent would (in the analyst's model) plan this 24-hour regimen where money is earned to purchase commodities instantly consumed, followed by leisure/rest for the remainder of the day.[9]

Leisure/rest is accordingly, and necessarily, absent as an independent variable in orthodox neoclassical micro-economic theory (as, for example, in Becker's work, discussed below). A way to exit this unrealistic condition is to identify utility exclusively with all human activity—rather than partly with human activity and partly with commodities. The present paper, while retaining the keystone commodity-utility assumption to allow unification of neoclassical and Gossenian theory, is a step in this direction.

Unifying Consumption-Constraint. As has been noted, a prime objective in the present contribution is completion of Walras' competitive system in equilibrium by defining and adding or inserting consumption-duration by way of a fully-generalized constraint on the commodity utility function. It will be seen that the two distinctly different mathematical theories of economics—Gossenian and neoclassical—which have been in opposition since the late nineteenth century, are unified in a common formulation by this constraint.[10]

Since Walras' contribution, very nearly all articles on mathematical economics published in economics journals up to the present time have maintained the keystone (commodity-utility) assumption of neoclassical theory. In this regard, Gary Becker in his 1965 article "A theory of the allocation of time," in which he attempted "…the systematic incorporation of non-working time" into economic theory, adhered to the keystone assumption by entering commodities directly into the utility function. And concerning consumption-duration, Nicholas Georgescu-Roegen in his review of Becker's article concluded that the writer "…[disregarded] the duration necessary for consumption, as revealed by a note referring to a 'home' production function that includes no time at all." (1983, p. ci.) But did Becker consistently disregard the "duration necessary for consumption"? I have found it quite difficult to confirm or oppose the conclusion with certainty. For this reason the present temporal theory is not a direct extension of Becker's mathematical system, but rather a combining of existing neoclassical fundamentals with (mainstream-overlooked) consumption-duration, formulated and extended in accordance with his stated vision. Besides the technical attributes of this synthesis, it will be seen, as noted, that the

[9] Here we make the analytically reasonable assumption that the essential economic activities (labor and leisure/rest…plus vitally important consumption) are mutually exclusive. And here we need to keep in mind that while consumption-utility is always modeled in orthodox/traditional mathematical economics, the *time-for-consumption* has been ignored (again, in orthodox/traditional mathematical economics) in our university economics departments around the world (as evident in the literature).

[10] An overview of the development of the Gossen equation from Gossen's book in 1854 through unification with neoclassical mathematical theory in 2011 is available in Chamberlain 2012 (Originally 2010).

historically significant unification of the two branches of mathematical economics—neoclassical and Gossenian, on separate paths since the 1870s—is also achieved.

In the developments to follow the first step is to define and explain the unified, fully-temporal mathematical system. Then a relationship between long-run utility discounting and the marginal productivities of labor and capital is derived, serving to promote the potentially fruitful integration of micro/macro economic theory. Additional progress along this line is completion of Walras' theory of the competitive economy in equilibrium by accounting for the missing terms in his short-term (e.g., same day) input/output substitution relations followed by extending these relations over the long-run interval accounting for intertemporal discounting.

II. Unified Fully-Temporal Economic System

A significant irony in the progression of neoclassical economics is that the highly mathematical subject had its origin in the (nevertheless important) contributions of a scholar with a limited knowledge of mathematics. The historian William Jaffe recorded his own conclusion (1973, p. 133) as to "Why did Leon Walras have to wait until [Paul] Piccard [1844-1920], a professor of mechanics and not an economist at all, pointed the way?" Jaffe's answer to his rhetorical question was Walras' "…inadequate training in mathematics."

Although Walras had taken special courses in preparation for entering the Polytechnic School of Paris, he twice failed entrance examinations before acceptance to the Paris School of Mines. At this point "…he still knew nothing about the extreme values of functions." (See Jaffe's footnote transcribed below.[11]) And while enrolled in the School of Mines "…he was a student in name only—in order to give himself an acceptable status in the eyes of his parents while he dabbled in novel writing." From this unpromising background emerged neoclassical economics, a paradigm that has yet to measure up to what is arguably the single most important academic subject in our modern world.

Paradigms are "sticky" and, for various reasons, universities have preserved and promoted Walrasian neoclassical economics over the past century despite an epistemological problem at its foundation—namely, whether instantly measured *feeling-state* (Dolan 2002) or time-integrated feeling-state (subjective utility) should be the primary concept (see Georgescu-Roegen's introduction to the English translation of Hermann Gossen's book *The laws of human relations and the rules of human action derived therefrom*, [1854] 1983, lxxxi). With the present paper this issue is closer to resolution.

[11] "This is evident from a manuscript dated '4 December 1853' which I have recently (1966) identified as L.W.'s, though it was misclassified in the Fonds Walras among A.W.'s under the mark F.W. VI. Written at the time he was studying for his entrance examinations, it was an elaborate exercise in "The Decomposition of Rational Functions into Partial Fractions," but shows no clear understanding of the theory of maxima and minima."

More to the point, it is shown herein that a single mathematical system bridges the divide between neoclassical theory and Gossenian theory. This is accomplished on the basis of a new constraint on the utility function, a constraint that relates consumed-amount of a commodity to its general dependence on the consumption-duration (i.e., a general dependence where both consumption-rate and consumption-duration vary at the margin).

An immediate question concerns Becker's theory of time in economics, and, as noted earlier, why his theory does not give a complete accounting of time in the subject. The most important reason, as indicated by Georgescu-Roegen in 1983 and Steedman in 2001, is that Becker's theory ignores or overlooks consumption-duration. In other words, while Becker acknowledged labor (work) duration and the activity-durations attending his '*productive*' *consumption* activities,[12] he "apparently" disregarded the consumption-duration of his "basic" commodities Z_i—which commodities (intended for household consumption and not trade) resulted from the short-term combining of '*productive*' *consumption* with market commodities. I write "apparently" because considerable effort has failed to firmly establish this conclusion—notwithstanding Georgescu-Roegen's observation (noted earlier) that Becker defined a basic commodity which was exclusive of time.

We are left with some uncertainty whether Becker did, in fact, "disregard the duration necessary for consumption." In a sense this uncertainty doesn't matter: either he disregarded the consumption-duration, or at least sometimes did so, or he didn't. If he disregarded consumption-duration, always or only sometimes, then the theory is incomplete. And if his home production functions did include the consumption-duration then the utility of any given consumption is counted twice, once by way of the market commodities within the basic commodity and again by way of the basic commodity's consumption-duration. In any case there is a problem at the foundation.

A second reason why Becker's theory cannot be fully temporal is that he assumed, in effect, that the individual experiences no feeling while working, as evidenced by the absence of work-duration in the utility function. The absence of work utility in his theory is puzzling, inasmuch as accounting for the utility (and disutility) of labor is established in orthodox theory.[13,14]

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[12] Becker's '*productive' consumption* activities are significant regarding long-run theory as well as short-term theory, in that *recurrence-of-wants* is recognized, albeit indirectly, when he justified the vector character of time inputs by "…the hours used during the day or on weekdays may be distinguished from those used at night or on weekends. (p. 495)." Recurrence-of-wants, an obvious property or aspect of human consumptive behavior, is implied—but it is in fact required, for the theory to make sense over extended time. In the present work, intertemporal recurrence-of-wants is crucially germane to the idea of periodic-equilibrium (as this law is to equilibrium theory in general).

[13] Work frequently engages or brings positive utility that typically becomes negative at the margin in any given day. We will accordingly adopt "utility" in the general sense rather than "disutility" in discussing the individual's psychosomatic experience while working.

[14] "Orthodox neoclassical economics" is defined as mathematical theory based on subjective utility wherein commodities (goods and services) always directly enter the utility function with work-duration as an optional

A third reason why Becker's theory is not fully temporal is that it applies only to a short-term interval—a single week, in his formulation—whereas a complete theory would account for time to the long-run intertemporal horizon. In this regard, long-run (intertemporal) discounting of expected utility is a necessary condition for an equilibrium formulation that is both realistic and determinate. As a particular, long-run discounting is an important factor in determining the relative magnitudes of the marginal productivities of labor and capital (as shown below).

Notwithstanding these shortcomings, the essential attribute of Becker's system—the recognition of '*productive*' *consumption* duration on the consumption side of the expectational plan—is a meaningful advance.

The objective in the following development is extension of neoclassical micro-economic theory to account for time spent on consumptive activity—in addition to time spent on productive activity (established in standard theory) and leisure/rest (which Walras indirectly recognized but is now ignored in neoclassical theory). This objective is achieved by starting with traditional theory, then progressing to the Becker theory, and finally arriving at the complete *human-activity based* intertemporal theory.[15]

A. Traditional Neoclassical System

Becker began his mathematical development by writing what he called the "traditional neoclassical system of household economic behavior", consisting of maximizing the utility function[16] (the short-term brackets $\langle --- \rangle_{\text{Day-t}}$ are omitted).

$$U = \bar{U}^x(\mathbf{x}), \tag{1}$$

subject to the budget constraint.

$$I = \mathbf{px},$$

where **x** represents market commodities, **p** their prices, and I is income.[17] It is seen that activity-durations are absent in this (Becker's) statement of traditional theory. We should note however that many neoclassical economists have recognized activity-durations in mathematical theory—beginning with

variable also entering the utility function. (Leisure/rest-duration cannot enter the utility function in orthodox economics as an independent or stand-alone variable for the reasons given earlier in the article.)

[15] Becker began with the "traditional" neoclassical system, and we do so here to provide continuity with his mathematical development.

[16] Utility is understood to be cardinal in my work. The ordinal utility character rose to prominence in the early twentieth century in the belief that cardinal utility was not necessary in modeling economic choice. But accounting for investment-risk discounting (and discounting in general), among other reasons, requires cardinal utility. (See Chamberlain 2003/4 and Leontief 1966, p. 26.) In this regard as an overarching consideration, utility is indisputably cardinal inasmuch as the variable is time-integrated feeling state, where both time and feeling-state are directly measurable (cardinal).

[17] Throughout the paper, vector symbols are in bold type and scalar symbols are in italics.

Walras and Jevons—and so the foregoing "traditional" system may have insufficient scope. In any case Becker observed that mathematical economists have not totally overlooked the importance of human activity in economic function, but rather that he or she "…simply [assumes] that the amount of time used at an activity is a direct function of the amount of goods consumed" (Becker 1965, p. 495fn). It was Becker's intent to substantively formulate human activity in economic theory, and we will now present his theory in moving toward the fully temporal theory.

B. Becker's '*Productive*' *Consumption* System

Becker departed from the foregoing elementary or essential neoclassical system by explicitly recognizing work-duration and also by introducing the innovative '*productive*' *consumption* durations which combine with commodities to yield "basic commodities" (Z_i; i=1,m). However, the utility that the individual expects to experience while working was disregarded. Furthermore, consumption-duration was not explicitly or conclusively evident in his mathematical system.

With the foregoing considerations in mind, Becker's baseline system (1) becomes his "Theory of the allocation of time" (1965) (the short-term brackets $\langle\text{---}\rangle_{\text{Day-t}}$ are omitted): [18]

$$U = \bar{U}^x(\mathbf{x}) + \underline{U}^A(\mathbf{T^A}) \quad ; \qquad \mathbf{x} = \sum \mathbf{x_i} \;\text{ and }\; \mathbf{T^A} = \sum \mathbf{T_i}\,,$$

subject to the constraints

$$\mathbf{p\,x} = I = V + \varpi T^W \;\; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \;\text{(Budget)}$$

$$T = T^W + \mathbf{I_n T^A} \;\; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \;\text{(Time)}$$

$$Z_i = f_i\,(\mathbf{x_i}, \mathbf{T_i}) \quad ; \; \text{i} = 1, \text{m} \qquad . \; . \; . \; . \; . \; . \; . \; . \;\text{(Basic commodities)}$$

The advances of this system over (1) consist of: introducing the (aggregate) '*productive*' *consumption* durations vector $\mathbf{T^A}$ which enters both the "additive" utility function $U^A(\mathbf{T^A})$ and the now recognized time-constraint; introducing work-duration T^W which enters the budget and time-constraints but not the utility function; and introducing the "basic commodities" Z_i which are obtained by combining commodities $\mathbf{x_i}$ with the corresponding '*productive*' *consumption* durations $\mathbf{T_i}$.

Examples of Becker's basic commodities are given in his paper—e.g., "…the seeing of a play, which depends on the input of actors, script, theatre, and the playgoer's time." But the example seems to count the utility of actively attending the play—where one $\mathbf{T_i}$ element would be "the playgoer's time"—along with the utility of the corresponding commodity which enters the utility function—where at least

[18] Becker wrote in terms of the economic behavior of "households"—as for example, when "..households maximize utility functions." (p. 495). In the present work we formulate the expectational plan of the individual.

one element of $\mathbf{x_i}$ is in some way (but always by *imputation*) inculcated or instilled with the satisfaction of attending the play. This suggests utility "double counting," which, were this to exist in his formulation, would undermine the theory. As another concern, it could be considered problematic to represent in Z_i the basic commodity's own consumption-duration (i.e., as an explicit variable). These considerations leave us uncertain whether the above example of seeing a play (and other examples in Becker's paper) truly represents his theory.

Note that the scalar time-constraint departs from Becker's vector time-constraint. In explaining this difference we may first note that the vector character of Becker's time constraint results from dividing his single-period (one week) into subintervals, each with its own set of durations.[19] In the present formulation we set aside the sub-intervals in favor of one interval—the single day—irrespective of weekends, second shift, etc. This simplification should not alter Becker's system in any way that affects his basic theory.

C. Fully Temporal System

The expectational plan extends from the real-time here-and-now datum forward to the individual's long-run intertemporal horizon. Within this overall expected-time interval reside shorter expected intervals which we call (as noted) short-term. How does short-term differ from long-run? In answering this question we first make an assumption that promises to be useful in certain kinds of economic analysis. The assumption is that the individual's expected long-run regimen is periodic: whatever his or her expected sequence of activities and all other expected activities may be (i.e., of other people, machines, and economic function of every kind, realistic or not), his or her short-term economic function at the beginning of the expected long-run is repeated to the very extreme of the intertemporal interval.[20] The short-term is then the basic period within the expected long-run interval. A candidate short-term interval is the single day, and this interval is adopted herein.

We will see that the periodicity assumption leads to new and potentially important insights into economic behavior. But what is the conceptual basis for the periodicity assumption—the condition should not be proposed without justification. The conceptual basis is this: that the single day d is postulated

[19] Becker wrote that time inputs comprise a vector because "…the hours used during the day or on weekdays may be distinguished from those used at night or on weekends. (p. 495)" But a basic commodity's time inputs in any given period (eg, a day) already comprise a vector, and do not require different aspects of time—eg, weekdays versus weekends—to engender the vector quality.

[20] Expected activity is "repeated" to first order. Departures from periodicity arise at higher orders. Note that this consideration de-emphasizes the problematic "representative-agent" assumption, whereby macro-behavior is simply the scaling-up of micro (agent) behavior. Nevertheless, agent behavior is germane to macroeconomics, just as particle-physics is certainly germane to large-scale physics. It is appropriate in this regard that economists cooperate with scholars of the sister social sciences (e.g., sociology and social-psychology) in modeling micro and macro behavior leading to more meaningful policy recommendations.

much shorter than the *utility-discount* characteristic time, which is postulated much shorter than the long-run characteristic time (Λ^{-1})—where this conceptual basis is called *nested-characteristic-times*.[21] And so, when the condition of nested-characteristic-times is postulated, long-run changes are negligible due to utility discounting ($\varepsilon_{\beta\text{-}\Lambda} = \Lambda/\beta << 1$) and short-term behavior is short enough to ignore the variation of utility-discounting ($\varepsilon_{d\text{-}\beta} = \beta\, d << 1$).[22]

The resulting long-run periodicity can be a useful mathematical assumption, one for example that allows insight into the relationship between labor and capital marginal productivities in the face of investment risk. Note, however, that the individual may see change ahead—perhaps even of a tragic nature—but expected utility is discounted so heavily that, in the economic sense, he or she doesn't care.

In the following developments we begin by formulating the complete short-term system for the single day and then address the long-run system. Regarding the long-run system, we will assume, as discussed above, that the system is in periodic-equilibrium—that is, each day in the system is identical to what transpired the day before and what will happen the following day (i.e., to first order). In the formulations it is understood that the utility function is maximized subject to utility-discounting and the imposed constraints.

We should emphasize at this point another important departure of the Gossenian approach from the approach of traditional neoclassical economics:

> *In neoclassical theory we start by assuming agents are "rational", as observed by Becker (1962),*
>
> > *"Now everyone more or less agrees that rational behavior simply implies consistent maximization of a well-ordered function, such as a utility or profit function." (page 1)*
>
> *and*
>
> > *"…since rational behavior is now taken to signify maximization of a consistent and transitive function." (Page 3)*
>
> *The problem here is—"<u>perfectly</u> rational"? <u>Yes!</u> –to the absolute exclusion and omission of "action or opinion given through inadequate use of reason, emotional distress, or cognitive deficiency"…which (categorically) excludes the entire human population, since* <u>no one</u> is perfectly rational. *This neoclassical assumption, resulting largely by default*

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[21] It is understood that the company's departure from steady-state performance is of the same order as that of the macro-economy.

[22] The condition $\varepsilon_{d\text{-}\beta} = \beta\, d << 1$ is a reasonable assumption for real economic behavior—fortunately, because it sets aside the considerable difficulty attending uneven discounting within the single day.

from the 1870s "wrong turn", is opposite the (highly successful) approach of applied-physics/engineering, wherein we start with comprehensive theory (e.g., the laws of fluid mechanics and thermodynamics) and then justify simplifying and defining assumptions.

The Gossenian approach, as noted earlier, follows (rigorously) the approach of applied-physics/engineering by starting with the comprehensive (as best we know) and going from there. But does the Gossenian approach cover the "irrational"—for examples: informational issues, heuristics and biases, intertemporal choice, and decision context (from Knoll (2010)). Yes…through the tried and true approach of applied-physics/engineering. And in this individuals do maximize in expectational planning—even when the intent may be injurious and counter-productive.

(a) Short-Term (Single Day) System.

The short-term system represents economic behavior in its most elementary form. Despite its elementary character, or more correctly because of it, the system is of crucial importance. Indeed, without a satisfactory formulation of short-term relationships, understanding and modeling the interaction of human activity and productive capital over the long-run must be similarly inadequate.

Becker's '*productive*' *consumption* model of the individual's economic behavior is not far removed from the complete short-term system. What is missing? Well, the individual's time at work is explicitly modeled, but his or her feeling-state on the job is disregarded: the variable does not enter the utility function. But arguably considerably more important is the (apparent) disregard of consumption-duration (partially or completely)—and this variable is accordingly given substantially greater attention in the next few paragraphs.

Work Utility. In accounting for work utility, Becker's work-duration is retained but the variable now enters the utility function. In this regard, recognizing the utility of work (and work disutility at and near the margin) is, of course, consistent with neoclassical orthodoxy: prominent neoclassical economists—as examples, Jevons (1871) and Ramsey (1928)—recognized the utility of work in their contributions.

Consumption-duration. Turning now to the duration-for-consumption, because Becker's theory on the consumption side is unclear—due to the question of whether consumption-duration is or is not disregarded, with its attending implications for utility double-counting—it is appropriate to retain only his progressive goal of "…the systematic incorporation of non-working time." (p. 495.) In the spirit of his intended advancement of neoclassical theory we will introduce or model consumption-duration, but without entering consumption-duration into the utility function. Instead, utility will, as usual, be directly identified with commodities (including the services of "capital goods such as automobiles and refrigerators") in accordance with orthodox neoclassical theory. Then an advanced consumption-constraint is applied to the neoclassical utility function, which constraint defines the *general* functional relationship between the amount consumed of a commodity and its consumption-duration.

The new constraint on the consumption utility function may be written (the single-day brackets $\langle\text{---}\rangle_{\text{Day-d}}$ are omitted)

$$x_k(T_k^C) = \int_0^{T_k^C} r_k(T_k^C, t)\, \mathrm{d}t = \int_0^{T_k^C} R_k(T_k^C)\, \mathrm{d}t = R_k(T_k^C)\, T_k^C \qquad (2)$$

where T_k^C is the consumption-duration of commodity k (i.e., of the quantity x_k) within the single-day, $r_k(T_k^C, t)$ is the instantaneous—or "instant"—rate-of-consumption of commodity k within the single-day from $t = 0$ to T_k^C , and $R_k(T_k^C)$ is the average rate-of-consumption (or *consumption-rate*) over the same interval (i.e., from $t = 0$ to T_k^C). It is seen that the amount consumed x_k is the product of consumption-duration T_k^C and its corresponding consumption-rate $R_k(T_k^C)$, a product which is essentially germane to the unification of neoclassical and Gossenian mathematical theory given herein.

It is seen that $r_k(T_k^C, t)$ "disappears" from (2) upon integration over the consumption-duration. This is an important aspect of mathematical economics, one that has its counterpart in thermodynamics. In standard (classical) thermodynamics, the true molecular character of matter is replaced by an assumed continuum which is modeled using analytic variables (e.g., pressure, density, temperature, viscosity, specific heat, etc.). In this "disappearance" of molecules, the functional relationships between variables are typically empirically determined (e.g., dependence of viscosity and specific heat on gas temperature). In economics, irregular consumption activities can be similarly modeled using analytic variables—i.e., modeled using average consumption-rates over their corresponding consumption-durations. As two particulars, (average) consumption-rates over their durations for given commodities can be measured and modeled, and on-the-job productivity versus (off-the-job) leisure/rest duration can be empirically formulated—where such functional relationships would be analytic within an activity-based methodology.

Equation (2) is not entirely new in the neoclassical literature. In their respective models, Linder (1970) and DeSerpa (1971) employed the constraint in its Gossenian mode where R_k is a specified constant, independent of T_k^C.[23] Steedman, in his extended investigation of consumption-over-time, also

[23] It is noteworthy that Gossen's contribution (1854) was not acknowledged, where this oversight has been customary in mainstream mathematical economics.

employed the Gossenian consumption constraint. In the *general* form of the constraint, however, both consumption-duration and consumption-rate vary at the margin.

We now investigate consumption-duration more closely and note that the infinitesimal marginal vector—whether neoclassical, Gossenian, or general—is always tangent to the corresponding consumption-duration curve, $x_k = R_k(T_k^C)\, T_k^C$. For the general case the differential of (2) is

$$\begin{aligned} \mathrm{d}x_k &= (T_k^C\, \mathrm{d}R_k / \mathrm{d}T_k^C)\, \mathrm{d}T_k^C + R_k\, \mathrm{d}T_k^C \\ &= T_k^C\, \mathrm{d}R_k + R_k\, \mathrm{d}T_k^C \end{aligned} \tag{3}$$

where we acknowledge that analytic dependence of R_k on T_k^C will naturally vary as the individual "explores" alternative activity scenarios.

In the two limit-cases to be addressed immediately below, one (Gossenian) will hold the consumption-rate invariant ($\mathrm{d}R_k \to 0$) and the second (neoclassical) will hold the consumption-duration invariant ($\mathrm{d}T_k^C \to 0$). Note, again, that the consumption-duration curve is understood to move and rotate as the individual converges on his or her optimized plan-of-action (including the "target" limit-cases).

Consider first the neoclassical case. Perhaps the closest an orthodox neoclassical economist has come to explicitly recognizing consumption-duration in an economic system is Frank Ramsey in his "A mathematical theory of saving" (1928), wherein he introduced consumption instant utility as the "..rate of utility of a rate of consumption x" (underline inserted). By "rate of consumption x" he meant amount of x consumed per unit time—i.e., consumed in a single second or single hour or single day. Each of these intervals—the single day, for example—is in effect a fixed consumption-duration, over which x is consumed.[24] Here the total consumption in a unit of time at a given (expected) time can vary (increase or decrease), but this occurs from an increase of rate and not duration—a property or condition of standard neoclassical mathematical economics that originated in the marginal-revolution contributions of Walras and Jevons. The differential of the neoclassical consumption-duration is accordingly zero, and (3) becomes

$$\begin{aligned} \mathrm{d}x_k &= T_k^C\, \mathrm{d}R_k + 0 \\ &= T_k^C\, \mathrm{d}R_k\,. \end{aligned}$$

We see that a marginal increase of x_k (as part of plan optimization) is due only to a differential increase of the average slope of the consumption-curve.

[24] In Ramsey's formulation consumption proceeded throughout the entire day, and was partly simultaneous or coincident with the duration-for-production. Working and eating (for example) at the same time does of course occur in real life, and Ramsey's model (incidentally, or without meaningful intent) represented the duality.

Now consider the Gossenian case. Here consumption-rate is fixed and variable marginal consumption only occurs through variation of consumption-duration. Equation (3) accordingly becomes

$$\begin{aligned} \mathrm{d}x_k &= 0 + R_k \ \mathrm{d}T_k^{\,C} \\ &= R_k \ \mathrm{d}T_k^{\,C} \end{aligned}$$

We may note here that Gossen assumed straight-line consumption-curves in his theory; in the general case (and in real economic life) consumption-curves will be nonlinear.

Besides its role in the consumption-constraint the consumption-duration variable is also inserted in the time constraint. This extension of neoclassical theory gives the individual's "utility calculus" a dependence on his or her consumption-duration, while preserving the neoclassical keystone (commodity-dependent utility) as an inviolable postulate.

Leisure/Rest. Leisure/rest—a more substantive or basic term than "leisure" alone, a term that recognizes the crucial importance of regenerative rest to economic function—is the remaining activity-duration to be added to traditional theory to achieve Becker's goal of accounting for non-working time in economics.[25,26] In adding this variable, leisure/rest-duration is simply introduced independently of Becker's system, in order to achieve maximum clarity of the developments to follow. Leisure/rest-duration directly enters the utility function. It also enters the time constraint, alongside the durations for work and consumption, where the three durations sum to the total available time in each day (24 hours, for the normal or usual regimen).

With this introduction, mathematical formulation of the short-term (one-day) system may be written (the single-day brackets $\langle \text{---} \rangle_{\text{Day-d}}$ are omitted)

$$U(T^W, \mathbf{x}, T^{L/R}) = \underline{U}^W(T^W) + \bar{U}^x(\mathbf{x}) + \underline{U}^{L/R}(T^{L/R}) \qquad (4)$$

subject to the corresponding set of constraints

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[25] Becker's non-working *'productive' consumption* durations can be accommodated in the present fully temporal theory.

[26] It can be argued, as a real-life consideration, that leisure/rest typically involves the consumption of one or more goods—where, in the case of rest, the sleeping pill and use of a bed are examples. But since the "activity" is of course vitally important to the individual's economic function and behavior, leisure/rest-duration should have an explicit representation in economic theory. In this regard, some writers (Geogescu-Roegen, as an example, and the present writer) have considered modeling leisure/rest as mathematically distinct (as herein, to help the development and discussion). An alternative, more rigorously correct, is to model leisure/rest as another form of consumption, but with appropriate and corresponding attributes and constraints to set it apart from pure or normal consumption. …Note in this regard that labor stands in relation to consumption in a similar way, in that one can, and frequently does, consume (e.g., eat) while working. (These aspects of economic theory only arise when the Gossen equation, representing generic *human behavior*, is applied to *economic behavior*, where production, consumption, and leisure/rest activities are explicitly modeled.)

$$\mathbf{p}\,\mathbf{x} = iS + \varpi T^{W} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Budget)}$$

$$T = T^{W} + \mathbf{I_n}\,\mathbf{T^C} + T^{L/R} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Time)}$$

$$\mathbf{x} = \mathbf{x}(\mathbf{T^C}) = [\boldsymbol{\kappa}^{\mathbf{C}}]\mathbf{T^C} \text{ with } [\boldsymbol{\kappa}^{\mathbf{C}}] = [[\mathbf{T^C}]^{-1}\mathbf{x}] \quad . \; . \; . \; . \; . \quad \text{(Consumption)}$$

where iS is substituted for V, looking to later developments, and assumed additive utility is represented by separate functions. The bracket terms are diagonal matrices.

In accordance with the introductory paragraph, the additional changes made to Becker's formulation are: (1) work-duration is inserted in the utility function; (2) consumption-duration, as an obvious component of "non-working time," is defined and entered in the time and consumption constraints (but not in the utility function), where this step is independent or exclusive of Becker's "…vector of time inputs used in producing the ith commodity" in order to avoid unnecessary complexity and possible confusion; (3) leisure/rest-duration, another "non-working" time, is defined and entered in the utility function and the time constraint, also independently of the "vector of time inputs."

Despite these changes (4) could still be indeterminate (because, as the root cause, commodities continue to enter the utility function). The solution would be determinate when consumption-duration is given or formulated in the constraints—for example when the lunch hour at work is specified. And the solution would be determinate when the consumption-rate is similarly given—for example when an automobile is rented by-the-hour. But when both consumption-duration and consumption-rate are determined within the model (that is, by the agent and not the analyst)—as would occur in the home-consumption of food and beverages as a prime example—an indeterminate condition will exist. Indeed, for a positive-definite leisure/rest instant-utility, consumption-duration would vanish (consumption rate would become infinite and its duration infinitesimal) in maximizing utility across any given day, there being no analytic reason for consumption-duration not to vanish. …The feasible recourse here is to set aside the commodity utility assumption in favor of utility exclusively identified with consumptive activity.

Retention of the keystone commodity-utility assumption, $\bar{U}^{x}(\mathbf{x})$, does not negate the value of (4) in preparing useful models. The analyst can, in this regard, specify the consumption-duration curves, $x_k(T_k^{\,C})$—in situations where the curves are normally determined expectationally—using empirical data and/or separate analyses. This approach has been successfully used in applied physics.

At the risk of repetition, additional discussion of the complete short-term formulation is provided—looking ahead to completing the neoclassical input/output substitution relations.

Consider first the utility function. Most important from the neoclassical perspective is the requirement that utility retain a functional dependence on the commodity vector **x**. This keystone of neoclassical economics, dating from the work of Jevons and Walras in the early 1870s marginal revolution, effectively precludes or bans the direct identification of utility with consumption-duration. Why? Since utility is already identified with commodities that are consumed, to additionally assign utility to consumption-duration adds utility/satisfaction from double counting.

This is a subject at the frontier, and disagreement is inevitable. As examples, both Linder (1970) and DeSerpa (1971) entered commodities and their consumption-durations directly and concurrently into the utility function. This must be utility double-counting. There is no utility double-counting from entering either x_k or T_k^C in the utility function. The error is in retaining x_k when T_k^C is inserted, inasmuch as the satisfaction in x is pleasure integrated over consumption-duration. Because consumption-duration T_k^C has been inserted in the utility function, retaining x_k with its implicit consumption-duration is necessarily redundant.

Besides commodities that are routinely consumed there are "mere possession" or "just-being-there" goods which are profoundly different. "Just-being-there" goods (Steedman, 2001, p. 77-8) enter the utility function without corresponding consumption-durations, whether explicit or simply assumed. They are out-of-sight goods providing pleasure from mere ownership—"pleasure from owning [paintings, rare books, etc.] that are never looked at". But, as noted above, any economic-good that the theorist inserts in the utility function is modeled as providing satisfaction, and satisfaction (again) is pleasure integrated over time. This consideration immediately proves that inserting "just-being-there" goods in the utility function without corresponding consumption-durations explicitly recognized in the constraints—or at least verbally acknowledged as such—is an incorrect idea. …If one insists on inserting "mere possession" goods in the utility-function, then finite intertemporal time (at least in basic theory) for appreciative reflection is categorically important.

Returning now to the main discussion, work as an explicit variable in utility functions is another concern. The subjective utility of work has been recognized in neoclassical theory since its beginning in the marginal revolution. This begs the question as to why we have different treatments of consumption utility versus work utility in standard mathematical economics. The answer is that the keystone assumption only excludes consumption-duration from the utility function (again, to avoid utility double-counting) and leaves the door open for work-duration.

The introduction of leisure/rest into the short-term formulation (that is, in the utility function and time constraint) is also straightforward, inasmuch as this duration has been part of economic theory for over 135 years. What is analytically new is extension of the leisure concept to encompass rest. Here rest is certainly a profoundly important dimension of economic function, in that it both competes with work-duration and indirectly promotes work productivity through revitalization (see also Becker, p. 498). Leisure/rest could be considered an indispensable duration in any substantive and comprehensive system of economic behavior.

Having provided an overview of the utility function, attention can now turn to certain of its mathematical aspects. Consider first the commodity utility function $\bar{U}^x(\mathbf{x})$. This function recognizes the individual's subjective utility that he or she expects to experience from the consumption of the set of commodities **x**. For our present objectives, we assign a homogeneous one-to-one dependence of **x** on $\mathbf{T^C}$. This is accomplished by first defining a ratio of **x** to $\mathbf{T^C}$, and then multiplying the ratio by $\mathbf{T^C}$. Accordingly, $\mathbf{x} = [\boldsymbol{\kappa}^\mathbf{C}]\mathbf{T^C}$ where $[\boldsymbol{\kappa}^\mathbf{C}]$ is a diagonal matrix whose individual elements are the elements of **x** multiplied by the inverse of the corresponding elements of $\mathbf{T^C}$, i.e. (the single-day brackets $\langle \text{---} \rangle_{\text{Day-d}}$ are omitted),

$$[\boldsymbol{\kappa}^\mathbf{C}] = [[\mathbf{T^C}]^{-1}\mathbf{x}] = \begin{pmatrix} x_1/T_1^C & 0 & \cdots & 0 \\ 0 & x_2/T_2^C & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & x_n/T_n^C \end{pmatrix}$$

With this new formulation the present mathematical system unites two theories that have been in opposition for over 135 years.

We may note here that consumption-duration, along with appropriate assumptions, allows utility to be "mapped" from commodity-dependence to activity-dependence, and conversely. It is this combination of consumption-duration and (attending or associated) utility mapping that fully unites Gossenian and neoclassical microeconomics theory (and thereby carries over recent Gossenian advances). Moreover, utility-mapping brings a new empiricism into neoclassical theory inasmuch as $U(T^C)$ can be empirically determined by way of measurable feeling-state integrated over measurable time.

Limit-Cases. The deeper or more-general consumption-constraint is the central idea of this paper. As just noted above, the short-term system (4), including this consumption constraint, unites two prominent and distinctly different systems in the literature. One system (neoclassical) is Ramsey's "A mathematical theory of saving" (1928) and the second (Gossenian) is Gossen's *The laws of human relations and the rules of human action derived therefrom* [1854] (1983), as extended by Georgescu-

Roegen in 1983 to account for leisure and completed by the writer in 1993 (unpublished book; an overview and advanced developments are provided in 2003c).

In demonstrating the two limit-cases we first address the neoclassical utility function $U^x(\mathbf{x})$. Performing the derivative, where the $[\boldsymbol{\kappa}^\mathbf{C}]\mathbf{T}^\mathbf{C}$ is substituted for $\mathbf{x}$, allows[27] (the single-day brackets $\langle\text{---}\rangle_{\text{Day-d}}$ are omitted),

$$\mathrm{d}\bar{U}^x(\mathbf{x})/\mathrm{d}\mathbf{x} = \mathrm{d}\bar{U}^x(\mathbf{x}(\mathbf{T}^\mathbf{C}))/\mathrm{d}\mathbf{x} = \mathrm{d}\bar{U}^x([\boldsymbol{\kappa}^\mathbf{C}]\mathbf{T}^\mathbf{C})/\mathrm{d}\mathbf{x}$$

$$= \mathrm{d}\bar{U}^x/\mathrm{d}([\boldsymbol{\kappa}^\mathbf{C}]\mathbf{T}^\mathbf{C})\Big\{\Big(\partial([\boldsymbol{\kappa}^\mathbf{C}]\mathbf{T}^\mathbf{C})/\partial\mathbf{T}^\mathbf{C}\Big)\mathrm{d}\mathbf{T}^\mathbf{C}/\mathrm{d}\mathbf{x} + \Big(\partial([\boldsymbol{\kappa}^\mathbf{C}]\mathbf{T}^\mathbf{C})/\partial[\boldsymbol{\kappa}^\mathbf{C}]\Big)\mathrm{d}[\boldsymbol{\kappa}^\mathbf{C}]/\mathrm{d}\mathbf{x}\Big\}$$

$$= \mathrm{d}\bar{U}^x/\mathrm{d}\mathbf{x}\Big\{[\boldsymbol{\kappa}^\mathbf{C}]\mathrm{d}\mathbf{T}^\mathbf{C}/\mathrm{d}\mathbf{x} + \mathbf{T}^\mathbf{C}\mathrm{d}[\boldsymbol{\kappa}^\mathbf{C}]/\mathrm{d}\mathbf{x}\Big\}.$$

This expression shows that the utility derivative at each $\mathbf{x}$ consists of two parts—a first part (after Gossen) where the consumption-rate of each commodity (e.g., x_1/T_1^C) is constant with a variable consumption-duration (i.e., variable T_1^C) and a second part (after Walras and Jevons, and faithfully followed by Ramsey) as the mathematical opposite, where consumption-duration is constant and the consumption-rate is variable.

Consider the limit-cases. First note that leisure/rest-duration $T^{L/R}$ was neglected in both Ramsey's and Gossen's systems, and is accordingly neglected here. The neoclassical orthodoxy followed by Ramsey is then obtained in the limit $\mathrm{d}\mathbf{T}^\mathbf{C}/\mathrm{d}\mathbf{x} \to 0$. In this contraction consumption-duration is simply fixed and any change in the amount consumed is the result of consuming at a faster or slower pace.

Turning to the second limit-case, while Ramsey's consumption-duration is invariant Gossen in his theory fixed consumption-rate with the result that total consumption of a commodity in a given day is proportional only to its consumption-duration. In mathematical terms, $\mathrm{d}[\boldsymbol{\kappa}^\mathbf{C}]/\mathrm{d}\mathbf{x} \to 0$ in the limit. Gossen accounted for work-duration, as did Ramsey, but like Ramsey (as noted), he did not accommodate leisure/rest-duration.

Consumption-duration plays an important role in the theory of substitution—between commodities, and between a commodity and paid labor, within both short-term and long-run intervals—as will be addressed later in the paper.

Unified Theory. We've seen above how the same system (4) covers or includes two prominent approaches to modeling economic behavior which are diametrically opposed in their formulation of consumption—Ramsey (following Walras and Jevons, and the neoclassical orthodoxy) treated consumption-duration as a fixed parameter (the unit of analytic time, e.g. one hour or one day)

[27] The equation may be studied in the scalar form rather than vector form by considering the single commodity x_k.

with consumption-rate as the salient variable, while Gossen fixed consumption-rate with consumption-duration as the salient variable. Recall, in this regard, that the divergence along different paths began during the marginal revolution of the 1870s, when Jevons and Walras too-late discovered Gossen's (complementary) theory.

It is important to reiterate that neoclassical and Gossenian theory are the same regarding the physical consumption of a commodity—although consumption-duration is typically implicit or "understood" in neoclassical theory. The difference arises in what happens at the margin in the process of maximization—in neoclassical theory T_k^C is invariant while in Gossenian theory R_k is invariant. But the two orthodoxies are equally unrealistic in the critical matter of plan or system optimization. This may be seen in the special case where the consumed-amount is fixed, i.e. x = constant (which has relevance in the food-stamp economy, now involving approximately 15% of the US population). Using the identity $x_k = R_k \bullet T_k^C$ we may write, for x=constant, (the single-day brackets $\langle --- \rangle_{\text{Day-d}}$ are omitted)

$$dx_k = (\partial x_k/\partial R_k)dR_k + (\partial x_k/\partial T_k^C)dT_k^C$$
$$= \mathrm{s}T_k^C dR_k + R_k dT_k^C = 0$$

which holds along the T_k^C versus R_k parabola. Everywhere along this parabola x_k is constant, even though both T_k^C and R_k continually vary. It is immediately seen that neither orthodox neoclassical theory (dT_k^C =0) nor Gossenian theory (dR_k =0) complies—for this important case both T_k^C and R_k must vary at the margin.

(b) Long-Run (Multi-Day) System.

The long-run (in periodic-equilibrium) is more than the short-term for each day placed end-to-end. Consider the relationship between labor and capital. Autonomic and investment-risk discounting of marginal utility, among other modes of discounting, progressively devalues the benefit expected by the individual from invested marginal labor disutility in Day-0. This, in turn—depending on the scale or degree of discounting—serves to determine the intensity of the indirect (capital-producing) labor in relation to the marginal productivities of direct (commodity-producing) labor and capital. Discounting also affects consumption and saving—but these behavioral dimensions will not be addressed below.

We will, as noted earlier, assume periodic-equilibrium in the following treatments. Autonomic discounting will be the initial basis for this development, followed by a second treatment based on investment-risk discounting.

Autonomic Discounting.[28] We proceed by several steps in extending the foregoing short-term system into a long-run system accounting for autonomic discounting. The first step is to propose that (4) represents not only the first day but every day, in all of its particulars except utility discounting, throughout the long-run. We accordingly obtain

$$U = \sum_{\text{d=0, n}} \left\langle \underline{U}^{W}(T^{W}) + \bar{U}^{x}(\mathbf{x}) + \underline{U}^{L/R}(T^{L/R}) \} \right\rangle_{\text{Day-d}} .$$

where we apply "d" to represent the single day within the multi-day long-run interval. All short-term days in the assumed periodic-equilibrium are subject to the utility constraints applied in the first day (Day-0).

As noted, we assume periodic-equilibrium over the long-run. It accordingly follows that the total long-run utility U is n times the single-day utility, but there are two difficulties: (1) the long-run is open-ended (i.e., without discounting); and (2) for a positive interest rate, equilibrium cannot be achieved due to ever-growing saving attending postponed consumption, where "…future consumption would approach satiety and saving-income ratios would rise close to unity" (Blaug [1968] 1978, p. 529). These difficulties can be resolved by inserting the autonomic discount parameter λ_d in the utility function, giving

$$U = \sum_{\text{d=0, n}} \left\langle \lambda \{ \underline{U}^{W}(T^{W}) + \bar{U}^{x}(\mathbf{x}) + \underline{U}^{L/R}(T^{L/R}) \} \right\rangle_{\text{Day-d}} + \left(o[\delta(\varepsilon_{d\text{-}\beta})] + o[\delta(\varepsilon_{\beta\text{-}\Lambda})] \right) \qquad (5)$$

where the higher order terms will vanish when $\varepsilon_{d\text{-}\beta}$ and $\varepsilon_{\beta\text{-}\Lambda} \to 0$. Note that λ_d is typically assigned an exponentially declining value (for convenience), but could have a non-exponential character (e.g., hyperbolic) more in line with empirical results.[29] To (5) we apply the constraints for each day d (the single-day brackets $\langle \text{---} \rangle_{\text{Day-d}}$ are omitted):

$$\mathbf{p}\,\mathbf{x} = iS + \varpi T^{W} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Budget)}$$

$$T = T^{W} + \mathbf{I_n}\,\mathbf{T^C} + T_d^{L/R} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Time)}$$

$$T = T^{W} + \mathbf{I_n}\,\mathbf{T^C} + T^{L/R} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Time)}$$

$$\mathbf{x} = \mathbf{x}\,(\mathbf{T^C}) = [\boldsymbol{\kappa}^{\mathbf{C}}]\mathbf{T^C} \;\; \text{with } [\boldsymbol{\kappa}^{\mathbf{C}}] = [[\mathbf{T^C}]^{-1}\mathbf{x}] \, . \; . \; . \; . \quad \text{(Consumption)}$$

where recall that the constraints are identical for all days in the periodic economic system.

But there is still a problem: the problem is that we have not accounted for the relationship between long-run discounting and productive labor/capital. In other words, we need to address the

[28] The individual may have some control over "autonomic" discounting. (See Ifcher and Zarghamee (2011))

[29] Note that while λ_d has always been dimensionless in economic theory, the parameter could be assigned the dimension $[\text{TIME}]^{-1}$ which would have the individual choose among candidate plans on the basis of subjective assessment (i.e., decision based on the most positive—or least negative—feeling state). (This prospect has been addressed in the literature—by Shackle (1958), Chamberlain (2003c), et. al.) We will, however, retain λ_d as a dimensionless parameter in accordance with neoclassical orthodoxy.

relationship between long-run discounting and the (marginal) productivities of labor and capital. This relationship is crucial to both short-term and long-run economics, and resides at the core or heart of economic function. As a particular, a vital factor affecting the expected magnitudes of labor-capital cooperation over the long-run expectation is missing when utility discounting is overlooked.

We will formulate the long-run relationship between λ_d and labor/capital function as simply as possible. First imagine a manufacturing company director—an overachiever working seven days a week—whose responsibility would normally be to maximize profits, and thereby directly determine her own compensation, in some proportion. But the general economic circumstances are such—perhaps due to a universally expected period of stagnant macroeconomic performance—where the corporate board expects only manufacturing efficiency. Additionally, she does not expect to save. In any case, her motivation rests ultimately with maximizing personal subjective utility.

We postulate a constant negative utility per unit time (instant disutility) on the job—i.e., $\mathrm{d}\underline{U}^W(T^W)/\mathrm{d}T^W$ has the same invariant (and negative) magnitude whether the director is working to preserve the total of productive-capital (in the morning) or working to produce the total of a marketable commodity (in the afternoon). We also postulate that her leisure/rest instant utility at the end of the day is both positive and invariant.[30] It remains to postulate how the utility of consumption is simplified.

We simplify the director's utility of consumption by postulating that her own personal output demand is fixed, say $\mathbf{x} = \mathbf{x_0}$, impervious to any new information or influences entering the intertemporal calculus. This postulate, along with no expected saving, means that the director expects to maximize managerial efficiency per se, as expected by the board of directors. (The purpose of this postulate—consumption inelasticity—is to allow us to study or analyze economic function on the production side of the intertemporal system without having to simultaneously account for what happens on the consumption side.)

With these three (utility) postulates, the director in each day expects to first (managerially) produce productive capital[31] at an invariant and negative level of instant utility; she then expects to (managerially) manufacture the company's output at the same level of (personal) instant utility; then, at home, to consume a very nearly fixed budget of commodities (purchased with her substantially invariant compensation); and finally to rest at an invariant and positive instant-utility for the remainder of the day—with this regimen expectedly repeated each day to her intertemporal horizon.

[30] We make this assumption as a convenience, inasmuch as the instant utility of leisure/rest will expectedly vary with duration (i.e., in magnitude and sense) depending on the individual's psychosomatic state—which significantly depends on life-experience (concerning work, family, health, etc.).
[31] Productive-capital consists of the director's skills and knowledge along with the company's capital of diverse kinds for producing goods and new productive capital.

On the basis of this expectedly periodic long-run regimen we may now formulate the relationship connecting autonomic discounting λ_d with: (a) the marginal productivity of the director's indirect labor (in producing productive-capital); (b) the marginal productivity of the director's direct labor (in producing output commodities); and (c) the combined (director and company) marginal productivity of productive-capital. In this formulation we adopt the exponential model of autonomic discounting. Accordingly,

$$\left\{ mp^{L1} \bullet mp^{cap\text{-}L2} / mp^{L2} \right\} \sum_{d=0,\,\infty} \lambda_0 \exp(-\beta_\lambda\, d)\, \exp([\pm]\Lambda\, d) = 1, \tag{6}$$

where d now has values zero to infinity (as a convenience), $\lambda_0 = 1$, and

mp^{L1} = marginal productivity of indirect labor in (managerially) producing productive-capital (the company's and her own);

$mp^{cap\text{-}L2}$ = marginal productivity of productive-capital (dimensionless) in producing the company's output, given the director's duration of direct labor;

mp^{L2} = marginal productivity of the director's duration of direct labor in (managerially) producing the company's output;

$\exp([\pm]\Lambda\, d)$ = variation of the system economic condition and any perturbation thereof—growing at a vanishing residual for [±] positive and similarly declining for [±] negative.

As an additionally note, Λ is the inverse of the expected characteristic time of long-run change (of the company and director's financial/economic condition), and also of the perturbation. This parameter is postulated to be much smaller (by over an order of magnitude) than β_λ in order to attain periodic-equilibrium. The exponential term could accordingly be dropped as having a negligible effect. It is seen in this model that productive-capital produces (in conjunction with labor) both itself and commodities.

Equation (6) is based on a labor/capital relationship—originating with Clark (1899) and applied by Lange (1936)—which is fundamental to the production of both capital and commodities in real-world economics. In applying this realistic relationship to derive (6) for the director in her corporate responsibilities it is postulated that personal and corporate capital continues or carries-over day-to-day. The company output-commodity, however, is entirely consumed in the same day it is produced. With the foregoing description and qualification, the following "walkthrough" of the long-run interval is provided to explain how (6) is obtained.

> In formulating (6) it is postulated that the director differentially increases her applied time in the morning of Day-0 by $\delta L1$, resulting in the perturbation $(mp^{L1}\ \delta L)\exp(-\Lambda\, d)$ with d=0. Later in the day, the director applies her time to the same capital, including the perturbation, to managerially produce the usual (day-to-day) output plus a differential output, the latter equaling $mp^{L1}\,\delta L1 \bullet mp^{cap\text{-}L2}$. Dividing this differential output by the director's marginal productivity mp^{L2} in (managerially) producing the output then yields a

differential reduction of her direct labor $\delta L2$. As the original capital perturbation (expectedly) endures unchanged, day-to-day, it continues to induce benefits in each day in terms of reduced direct labor and the corresponding reduced labor-disutility. Balancing the original $\delta L1$ (dis) utility against the beneficial $\delta L2$ (dis) utility reductions, summed and discounted, results in (6).

Autonomic discounting was introduced in the long-run utility summation to defeat or resolve an open-endedness in the individual's utility calculus—a condition characterized by unlimited saving attending perpetually postponed consumption. Then the resulting expression (5) was used to derive (as a new relationship in neoclassical economics) the effect of autonomic discounting on the marginal productivities of labor and capital (i.e., in the considered example, on the marginal productivity of the system's [company and director] productive-capital along with the marginal productivities of the director's [indirect and direct] labor in maximizing productive efficiency).

Using the above treatment of autonomic-discounting as the basis, a brief development is now provided demonstrating how labor and capital marginal productivities are also dependent on anticipated *investment-risk* in the economic system in periodic-equilibrium.

Investment-Risk Discounting. People at all levels account for risk in some manner in their investment plans. The contingencies in life, which burden the poor and middle-class more than the rich, properly call for different countermeasures (i.e., because of their inferior resources), which affect the relative levels of investments (in education/skills, health, and material capital), and which in turn distort, over extended time, how the national product is distributed across the population. Social and economic stability ultimately depends on how uneven investment risk affects this distribution.

While the relationship between investment risk and the marginal productivities of labor and capital can be derived from a more essential foundation (see 2003/4), we will in the following treatment simply adjust the earlier autonomic-discounting based relationship.

The first step is to give an example of expected (intertemporal) risk. Here we posit that the corporate director, considered above, expects a small but fixed chance per unit time (e.g., one day) that some contingency will effectively bankrupt and liquidate the company—i.e., end its existence. We designate this chance by β_I. The expected probability P that the company will still exist at a future time t is related to β_I by the differential equation

$$\mathrm{d}P/\mathrm{d}t = -\beta_I P$$

from which we obtain

$$P = P_0 \exp(-\beta_I t)$$

where $P_0 = 1.0$ at the present-time ($t = 0$) datum.

We now ask whether expected risk discounts long-run utility in a manner similar to autonomic discounting λ_d. The answer is yes—inasmuch as both $P(t)$ and $\lambda(t)$ serve to reduce expected utility at future time t to its present-time value. We may therefore apply $P_d = P_0 \exp(-\beta_I d)$ in conjunction with $\lambda_d = \lambda_0 \exp(-\beta_\lambda d)$ in (5) giving (substituting d for t in P_d)

$$U = \sum_{d=0,\infty} P_0 \exp(-\beta_I d)\, \lambda_0 \exp(-\beta_\lambda d) \left\langle \underline{U}^W (T^W) + \bar{U}^x (\mathbf{x}) + \underline{U}^{L/R}(T^{L/R}) \right\rangle_{\text{Day-d}} + \text{higher-order-terms}. \quad (7)$$

Note that this utility function is valid only for the case where the director expects a sudden and complete company failure, should this happen. The reason is that persistence or continuation of the (expected) aftermath over a finite time would negate the postulated periodic character of the long-run expectational plan.

With the recognition that investment risk is another aspect of intertemporal discounting, we may bring forward the marginal productivity versus autonomic discounting relationship (6) with $\exp(-[\beta_I d+\beta_\lambda d])$ substituted for $\lambda_0 \exp(-\beta_\lambda d)$:

$$\left\{ mp^{L1} \bullet mp^{cap\text{-}L2} / \, mp^{L2} \right\} \sum_{d=0,\infty} \exp(-[\beta_I d + \beta_\lambda d]) \exp([\pm]\Lambda d) = 1,$$

where both P_0 and λ_0 have value 1 and the variables are defined above (see (6)). Because we are specifically interested in investment-risk discounting, autonomic discounting is assumed relatively small and the equation becomes

$$\left\{ mp^{L1} \bullet mp^{cap\text{-}L2} / \, mp^{L2} \right\} \sum_{d=0,\infty} \exp(-\beta_I d) = 1,$$

where the negligibly small parameter Λ is also removed.

A profound difference between the above investment-risk based relationship and the earlier autonomic-discounting based relationship (6) is that anticipated investment-risk is circumstance- and plan-dependent, while autonomic discounting is substantially plan-independent. Should some change in circumstances effect a change in investment-risk, how would the director's regimen or performance change in managing both the company's productive capital and her own productive capital? This can be determined from a close inspection of the above relation.

We focus first on the investment-risk exponential $\exp(-\beta_I d)$. It is seen in the equation that a sudden (incremental) increase of β_I requires an equally sudden increase of the marginal productivity function in the braces. In this increase, mp^{L1} changes (increases) while $mp^{cap\text{-}L2}$ and mp^{L2} remain invariant. In explaining the invariance of $mp^{cap\text{-}L2}$ and mp^{L2}, first recall that output demand is postulated (very nearly) constant. This means that *L2* does not change since the level of capital on which it operates does not change. (The level of capital does change, of course, but beyond the intertemporal horizon—i.e., $\varepsilon_{\beta\text{-}\Lambda} = \Lambda/\beta$ remains <<1.) Given a fixed relationship between *L2* and mp^{L2}, for a constant level of capital, it follows that mp^{L2} must be invariant.

Turning now to the marginal productivity of capital, since both output and L2 are invariant it can be concluded that $mp^{cap\text{-}L2}$ is also invariant. And finally, under the condition of decreasing returns, the director's indirect duration *L1* must (immediately) decline to yield an increased marginal productivity.

What happens over extended real time? The productive capital of the director and the company she manages gradually declines, as obsolescence and deterioration take their toll. The director—and all company employees—must work harder to maintain the output on which their income depends. Eventually, increasing work-duration, eclipsing revitalizing leisure/rest, cannot be sustained, and the company fails.

III. Input and Output Substitutions in the Competitive Economy

A complete accounting of time in utility theory is of course necessary for a complete theory of the competitive economic system in equilibrium. Walras in this regard, in his profound deepening of economic theory (1874-77), did not produce a complete theory due to his neglect or oversight of the utility of work along with neglect of consumption-duration in the individual's intertemporal (short-term and long-run) planning. This had two consequences (at a minimum): (1) the utility of work was absent from his substitution equations; and (2) the value placed on leisure/rest was disregarded in the substitution of an output demanded (market commodity) for another output, or for an input supplied. (More to the point, as the individual in his or her expectation gains utility through increased consumption, utility is lost by the leisure/rest that is thereby eclipsed.) Walras did, however, account for the eclipse of leisure/rest by marginally increased work-duration in his substitution equations.

The short-term case (again, single-day)) is considered first followed by the long-run (multi-day) case which extends the single-day formulation to the discount horizon.

A. Short-Term (Same Day) Substitution

The several missing terms in Walras' original theory will be given in the following re-derivation of his substitution relations using the completed short-term system (4). Note that we will focus on the input/output equations—the reader is referred to Walras' original work, or Brems' (modern, 1986) restatement of this contribution, for the entire theory.

(a) Same-Day Substitution Between Commodities

Utility Equimarginality. At the solution point defined by (4) the constrained utility differential is zero. Accordingly, in a market substitution between two commodities, x_a and x_b,

$$
\begin{aligned}
dU &= (\partial U/\partial x_a)\,dx_a + \overbrace{(\partial U/\partial T^{L/R})\,(\partial T^{L/R}/\partial T_a^{C})(dT_a^{C}/dx_a)\,dx_a}^{A} \\
&\quad + (\partial U/\partial x_b)\,dx_b + \overbrace{(\partial U/\partial T^{L/R})\,(\partial T^{L/R}/\partial T_b^{C})\,(dT_b^{C}/dx_b)\,dx_b}^{B} \\
&= (\partial U/\partial x_a)\,dx_a - \overbrace{(\partial U/\partial T^{L/R})\,(1/[\kappa_a^{C} + T_a^{C}\,d\kappa_a^{C}/dT_a^{C}])\,dx_a}^{A} \\
&\quad + (\partial U/\partial x_b)\,dx_b - \overbrace{(\partial U/\partial T^{L/R})\,(1/[\kappa_b^{C} + T_b^{C}\,d\kappa_b^{C}/dT_b^{C}])\,dx_b}^{B} \\
&= 0
\end{aligned}
\tag{8}
$$

where x_a and x_b are different elements of the commodity vector **x** and T_a^C and T_b^C are their respective durations-for-consumption of the vector $\mathbf{T^C}$. κ_a^C and κ_b^C are elements of the facilitator diagonal matrix which connects commodity quantities to their consumption durations. $T^{L/R}$ is leisure/rest-duration. (Note that $\partial T^{L/R}/\partial T_a^C = -1$ and $\partial T^{L/R}/\partial T_b^C = -1$ from the time constraint.) The terms A and B are new—further discussed below.

Financial Parity. We additionally require that a (differential) substitution between x_a and x_b satisfy the budget constraint. Accordingly,

$$
\begin{aligned}
&(\partial(\mathbf{px})/\partial x_a)\,dx_a + (\partial(\mathbf{px})/\partial x_b)\,dx_b \\
&\quad = p_a\,dx_a + p_b\,dx_b \\
&\quad = 0
\end{aligned}
\tag{9}
$$

where p_a and p_b are the respective elements of the price vector **p**.

Combining (8) and (9) yields,

<u>Same-Day Substitution Between Two Commodities</u>:

$$
\begin{aligned}
dx_a/dx_b &= -\left\{(\partial U/\partial x_b) - \overbrace{(\partial U/\partial T^{L/R})\,(1/[\kappa_b^{C} + T_b^{C}\,d\kappa_b^{C}/dT_b^{C}])}^{A}\right\} \\
&\quad \Big/ \left\{(\partial U/\partial x_a) - \overbrace{(\partial U/\partial T^{L/R})\,(1/[\kappa_a^{C} + T_a^{C}\,d\kappa_a^{C}/dT_a^{C}])}^{B}\right\} \\
&= -\,p_b/\,p_a
\end{aligned}
\tag{10}
$$

The terms A and B are new to Walras' theory, and to modern theory that has evolved from the foundation Walras prepared. These terms represent the leisure/rest utility that is eclipsed or extinguished when the individual consumes an additional amount of a commodity. More to the point, when the individual expectedly purchases an additional differential of x_a (as an example) he or she increases the corresponding consumption duration by $dT_a^C = (dT_a^C/dx_a)\, dx_a = (1/[\kappa_a^C + T_a^C d\kappa_a^C/dT_a^C])\, dx_a$. This, in turn, produces a corresponding *negative* shift of the leisure/rest-duration, as required by the time constraint. The result is $-\,(\partial U/\partial T^{L/R})\,(1/[\kappa_a^C + T_a^C\, d\kappa_a^C/dT_a^C])$ in (10), a factor in the relationship of every individual to the market which he or she helps define.

(b) Same Day Substitution Between Paid-Labor and a Commodity

Utility Equimarginality. We again recognize dU = 0 at the solution point. Accordingly, in a constrained differential substitution between market commodity x_a and the individual's work input T^W

$$
\begin{aligned}
\mathrm{d}U &= (\partial U/\partial x_a)\,\mathrm{d}x_a + \overbrace{(\partial U/\partial T^{L/R})\,(\partial T^{L/R}/\partial T_a^C)(dT_a^C/dx_a)\,\mathrm{d}x_a}^{A} \\
&\quad + \overbrace{(\partial U/\partial T^W)}^{B}\,\mathrm{d}T^W + (\partial U/\partial T^{L/R})\,(\partial T^{L/R}/\partial T^W)\,\mathrm{d}T^W \\
&= (\partial U/\partial x_a)\,\mathrm{d}x_a \overbrace{-\,(\partial U/\partial T^{L/R})\,(1/[\kappa_a^C + T_a^C\, d\kappa_a^C/dT_a^C])\,\mathrm{d}x_a}^{A} \\
&\quad + \overbrace{(\partial U/\partial T^W)}^{B}\,\mathrm{d}T^W - (\partial U/\partial T^{L/R})\,\mathrm{d}T^W \\
&= 0
\end{aligned}
\tag{11}
$$

where (as before) x_a is an element of the commodity vector **x**, T_a^C is its corresponding consumption-duration, and κ_a^C is the interconnecting diagonal-matrix element. T^W is the scalar work-duration that the individual supplies to the market.

Financial Parity. We additionally require that a (differential) substitution or shift between x_a and T^W satisfy the budget constraint. Accordingly,

$$
\begin{aligned}
&(\partial(\mathbf{px})/\partial x_a)\, dx_a - (\partial(\varpi T^w)/\partial T^W)\, dT^W \\
&= p_a\, dx_a - \varpi\, dT^W \\
&= 0
\end{aligned}
\tag{12}
$$

where p_a , x_a , and T^W have been previously defined and ϖ is the individual's wage rate.

Combining (11) and (12) yields

Same-Day Substitution of Paid-Labor for a Commodity:

$$dx_a/dT^W = -\overbrace{\{(\partial U/\partial T^W) - (\partial U/\partial T^{L/R})\}}^{A} \Big/ \overbrace{\{(\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a^{\,C} + T_a^{\,C}\, d\kappa_a^{\,C}/dT_a^{\,C}])\}}^{B} = \varpi/p_a \qquad (13)$$

for the differential substitution between outputs demanded and inputs supplied. The terms A and B are new to input versus output substitution in the Walrasian competitive economy in equilibrium.

Should the individual supply more than one input to the market, the corresponding substitution equation between any two of the inputs is easily obtained as a straight-forward revision of (13).

B. Long-Run Substitution

In deriving the long-run substitution equations for transactions between present Day-0 and any specified Day-n assuming periodic-equilibrium and discounting, the procedure differs from the foregoing for the single short-term day by accounting for both the delay between commodity delivery on Day-0 and commodity receipt on Day-n along with the premium (based on discount) that attends the delay. It will be seen in the following that the substitution expression for invariant long-run utility ($dU = 0$), after discounting, in conjunction with the short-term budget constraints analytically confirms long-run discounting as a basis for the interest rate.

We address the solution point defined by (7) with autonomic discounting retained. Note, however, that conceptually complex investment discounting is suppressed in the development. Accordingly,

$$U = \sum_{d=0,\,\infty} P_0 \exp(-\beta_\lambda d)\, \big\langle \{ \underline{U}^W(T^W) + \bar{U}^x(\mathbf{x}) + \underline{U}^{L/R}(T^{L/R}) \} \big\rangle_{\text{Day-d}} + (\mathrm{o}[\delta(\varepsilon_{d\text{-}\beta})] + \mathrm{o}[\delta(\varepsilon_{\beta\text{-}\Lambda})]) \qquad (14)$$

where, as before, the higher order terms will vanish when $\varepsilon_{d\text{-}\beta}$ and $\varepsilon_{\beta\text{-}\Lambda} \rightarrow 0$, and the usual set of identical short-term constraints apply, i.e., (the single-day brackets $\langle\text{---}\rangle_{\text{Day-d}}$ are again omitted) .

$$\mathbf{p}\,\mathbf{x} = iS + \varpi_d\, T^W \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Budget)}$$

$$T = T^W + \mathbf{I_n}\,\mathbf{T^C} + T^{L/R} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Time)}$$

$$\mathbf{x} = [\boldsymbol{\kappa}^{\mathbf{C}}]\mathbf{T^C} \;\text{ with } [\boldsymbol{\kappa}^{\mathbf{C}}] = [[\mathbf{T^C}]^{-1}\mathbf{x}] \quad . \; . \; . \; . \; . \quad \text{(Consumption)}$$

Utility invariance is again required in the substitution, but now after accounting for utility discounting over the full long-run interval.

(a) Forward Substitution Between Two Commodities

Discounted-Utility Equimarginality. At the solution point $dU = 0$. Accordingly, in a considered market transaction where the individual expects to deliver or hand-over δx_a in Day-0 in exchange for δx_b to be received in Day-n,[32]

$$dU = \Big\langle \Big\{ (\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a^{\,C} + T_a^{\,C}\, d\kappa_a^{\,C}/dT_a^{\,C}]) \Big\}\, \mathrm{d}x_a \exp(-\beta_\lambda d) \Big\rangle_{\text{Day-0}}$$

$$+ \Big\langle \Big\{ (\partial U/\partial x_b) - (\partial U/\partial T^{L/R})\,(1/[\kappa_b^{\,C} + T_b^{\,C}\, d\kappa_b^{\,C}/dT_b^{\,C}]) \Big\}\, dx_b \exp(-\beta_\lambda d) \Big\rangle_{\text{Day-n}}$$

$$+ \big(\mathrm{o}[\delta(\varepsilon_{d\text{-}\beta})] + \mathrm{o}[\delta(\varepsilon_{\beta\text{-}\Lambda})] \big)$$

$$= 0$$

where, as before, the nomenclature $\langle \text{---} \rangle_{\text{Day-x}}$ signifies all terms within the brackets have day-x values. (Although unnecessary in the present periodic-equilibrium development—because expected daily plans are identical—designated-day brackets will encompass all Day-t terms to promote clarity.)

The expression may be rearranged to give

$$\langle dx_b \rangle_{\text{Day-n}} / \langle dx_a \rangle_{\text{Day-0}} = \Big\langle (\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a^{\,C} + T_a^{\,C}\, d\kappa_a^{\,C}/dT_a^{\,C}]) \Big\rangle_{\text{Day-0}}$$

$$\Big/ \Big\langle \Big\{ (\partial U/\partial x_b) - (\partial U/\partial T^{L/R})\,(1/[\kappa_b^{\,C} + T_b^{\,C}\, d\kappa_b^{\,C}/dT_b^{\,C}]) \Big\} \exp(-\beta_\lambda d) \Big\rangle_{\text{Day-n}} \qquad (15)$$

where the higher order terms are dropped as negligible as the limits $\varepsilon_{d\text{-}\beta} \to 0$ and $\varepsilon_{\beta\text{-}\Lambda} \to 0$ are approached.

It is seen in (15) that long-run discounting of expected utility requires, as compensation, an increased return on Day-n—by factor $\exp(\beta_\lambda n)$ over the delivery on Day-0.

An observation at this point is that (15) could not have been derived within standard neoclassical theory due to neglect of (variable) consumption-duration.

Financial Parity. Equation (15) shows that forward long-run substitution requires a premium to compensate for long-run discounting of utility. A similar premium is identified with interest in the (forward) substitution, as follows from the budget equation for periodic-equilibrium (see (14)). We recognize, in this regard, that a differential amount of commodity x_b to be received in Day-n with payment in terms of commodity x_a in Day-0 is increased by interest over the interval. Accordingly,

$$\langle dx_b \rangle_{\text{Day-n}} = \langle (\partial x_b/\partial x_a)\, dx_a \rangle_{\text{Day-0}} \exp(in)\ .$$

or

[32] As a clarification, the individual agrees to deliver an incremental δx_a to the market, or receive δx_a from the market, with the substitution completed at a later date.

$$\langle dx_b \rangle_{\text{Day-n}} / \langle dx_a \rangle_{\text{Day-0}} = -\,(p_a/p_b)\,\exp(in) \qquad (16)$$

after using $\partial x_b/\partial x_a = -\,p_a/p_b$ from the budget constraint and rearranging terms.

Substituting (16) into (15), with the higher-order terms vanishing in their limits, yields the expression for forward substitution of one commodity for another in a system in periodic-equilibrium:

Forward-Substitution Between Two Commodities:

$$\langle dx_b \rangle_{\text{Day-n}} / \langle dx_a \rangle_{\text{Day-0}} = -\,\langle \{ (\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a^C + T_a^C\, d\kappa_a^C/dT_a^C]) \} \rangle_{\text{Day-0}}$$
$$/ \langle \{ (\partial U/\partial x_b) - (\partial U/\partial T^{L/R})\,(1/[\kappa_b^C + T_b^C\, d\kappa_b^C/dT_b^C]) \} \exp(-\beta_\lambda\, d) \rangle_{\text{Day-n}} = -\,(p_a/p_b)\,\exp(i\,n). \qquad (17)$$

It is seen that the above expression is functionally equivalent to the same-day substitution except for the discount and interest exponentials.

Equation (17) permits deeper insight into the origin and meaning of interest. First we clear some of the complexity by simplifying (17) to the case of forward substitution of an identical commodity. The equation accordingly becomes

$$\langle dx_a \rangle_{\text{Day-n}} / \langle dx_a \rangle_{\text{Day-0}} = -\,\exp(\beta_\lambda n) = -\,\exp(in).$$

which yields

Equilibrium Interest Rate:

$$i = \beta_\lambda$$

showing that the (natural) interest rate and the autonomic discounting rate are identical when the individual's long-run intertemporal plan is in periodic-equilibrium, with all remaining modes of discounting suppressed.

(b) Forward Substitution of Paid-Labor for a Commodity

Discounted-Utility Equimarginality. In the considered expected transaction the individual works dT^W hours at the margin on Day-0 with payment in a market commodity on a later Day-n. Total long-run utility after discounting is expectedly unchanged in the transaction. Accordingly,

$$dU = \langle \{ (\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a^C + T_a^C\, d\kappa_a^C/dT_a^C])\, dx_a \exp(-\beta_\lambda d) \rangle_{\text{Day-n}}$$

$$+\ \langle \{ (\partial U/\partial T^W) - (\partial U/\partial T^{L/R}) \}\, dT^W \exp(-\beta_\lambda d) \rangle_{\text{Day-0}}$$

$$+ \left(\mathrm{o}[\delta(\varepsilon_{d\text{-}\beta})] + \mathrm{o}[\delta(\varepsilon_{\beta\text{-}\Lambda})] \right)$$

$$= 0$$

where, as before, the higher order terms vanish when $\varepsilon_{d\text{-}\beta}$ and $\varepsilon_{\beta\text{-}\Lambda} \to 0$. The expression is rearranged to give

$$\langle dx_b \rangle_{\text{Day-n}} / \langle dT^W \rangle_{\text{Day-0}} = -\left\langle \left\{ (\partial U/\partial T^W) - (\partial U/\partial T^{L/R}) \right\} \right\rangle_{\text{Day-0}} \Big/ \left\langle \left\{ (\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a{}^C + T_a{}^C\, d\kappa_a{}^C/dT_a{}^C]) \right\} \exp(-\beta_\lambda\, d) \right\rangle_{\text{Day-n}} \quad (18)$$

where the higher order terms are dropped as negligible.

It is seen in (18) that autonomically discounted utility requires a proportionately greater return on Day-n (by factor $\exp(\beta_I\, n)$) over the delivery on Day-0.

Financial Parity. As was the case regarding forward substitution of one market commodity for another under financial parity, the substitution of earned income at the margin for a future commodity entails an interest premium. With reference to the budget equation (see (14)), we recognize that a differential duration dT^W of paid labor in Day-0 is compensated for by a differential amount dx_a plus interest of a commodity to be received in Day-n. Accordingly,

$$\langle dx_a \rangle_{\text{Day-n}} = \langle (\partial x_a/\partial T^W)\, dT^W \rangle_{\text{Day-0}} \exp(i\, n)$$

or

$$\langle dx_a \rangle_{\text{Day-n}} / \langle dT^W \rangle_{\text{Day-0}} = (\varpi_d/p_a) \exp(i\, n) \quad (19)$$

after using $\partial x_a/\partial T_d{}^W = \varpi_d/p_a$ from the budget constraint and rearranging terms.

Inserting (19) into (18) yields the expression for forward substitution of paid-labor for a commodity at the margin in a system in periodic-equilibrium:

Forward Substitution of Paid-Labor for a Commodity:

$$\langle dx_a \rangle_{\text{Day-n}} / \langle dT^W \rangle_{\text{Day-0}} = -\left\langle \left\{ (\partial U/\partial T^W) - (\partial U/\partial T^{L/R}) \right\} \right\rangle_{\text{Day-0}}$$

$$\Big/ \left\langle \left\{ (\partial U/\partial x_a) - (\partial U/\partial T^{L/R})\,(1/[\kappa_a{}^C + T_a{}^C\, d\kappa_a{}^C/dT_a{}^C]) \right\} \exp(-\beta_\lambda\, d) \right\rangle_{\text{Day-n}} = (\varpi/p_a) \exp(in).$$

This expression, and the earlier equation (17) representing forward substitution of one commodity for another, are new to Walrasian mathematical theory, and similarly new to neoclassical theory which has ascended from the Walrassian foundation.

IV. CONCLUSION

The short-term and long-run intertemporal theory present herein is in accord with the neoclassical tradition, as advanced by Gary Becker's intent (1965) to account for "non-working time" in mathematical economics. Strict adherence (in the present work) to the neoclassical-economics keystone (i.e., commodity amounts directly enter the utility function) is perhaps most prominent in this concurrence.

This necessary retention of the keystone assumption in the new theory—as demanded by economics orthodoxy—requires (for the human activity-complete model) that the analyst specify the consumption-duration curves rather than have them occur as part of the solution. Not to do this entails the total eclipse of time-for-consumption in each day by time-for-leisure/rest, as the utility of the latter rises with increasing duration while the utility of the former is unchanged (because the keystone assumption excludes consumption-duration from the utility function).

In preparing the conceptual and mathematical advance to a fully long-run (intertemporal) theory, the development proceeded through four stages. In the first stage the "traditional" formulation was given, wherein the commodity-dependent utility function is constrained only by the budget equation. In the second stage a careful assessment of Gary Becker's "A Theory of the Allocation of Time" (1965) was performed, resulting in adoption of his vision for "…the systematic incorporation of non-working time" into economic theory but setting aside his "basic commodity" idea in order to reduce complexity and avoid possible confusion. In the third stage, which arrived at the complete short-term system, three aspects either missing or unclear in Becker's theory were introduced: (1) utility of work; (2) leisure/rest (entering the utility function and time constraint); and (3) consumption-duration, by way of a deeper and more general constraint which equates the amount consumed of a given commodity to the product (or multiplication) of its rate-of-consumption by its consumption-duration. (Consumption-duration enters the time constraint, but does not enter the utility function).

In the fourth stage, the fully temporal long-run system was obtained by applying intertemporal discounting over a continuous sequence of days to the discount horizon, with the recognition that labor/capital interaction or function is necessary to represent true economic behavior. Using this system in its assumed *periodic-equilibrium* condition, the relationship between long-run utility discounting (including investment-risk) and the marginal productivities of labor and capital, previously derived within the Gossenian school, was brought into neoclassical economics.

The study proceeded to using the deeper short-term system to complete the unfinished input/output substitution relations published by Leon Walras as part of his theory of the economic system in competitive equilibrium (1874-77). This same-day substitution was then extended to substitution within the long-run expectational plan accounting for discounting, and the formulation was used to equate the natural interest rate with the discount rate for an economic system in periodic equilibrium. These diverse advances, overarching micro- and macro-economics, point to the potential value of deeper micro-economic theory in understanding and stabilizing systemic function.

The 20+ year delay in recognition of the 1854 publication of Hermann Gossen's activity-based theory of mathematical economics until after the marginal revolution in the early-to-mid 1870s produced a schism or divide in economic theory that has lasted into the 21st century. If Jevons and Walras had earlier discovered Gossen's book they could have included consumption-duration in their theory as the crucial variable that it certainly is. This would have naturally led to the generalization of Gossen's consumption constraint (where only consumption-duration is varied at the margin) to account for variation of consumption-rate as well (the latter being the *basis-by-default* for optimization at-the-margin in orthodox neoclassical theory). Unification of the two fundamentally distinct mathematical theories in the present work serves to end the 135 year divide.

TERM DEFINITIONS

Consumption-curve = defined by

$$x_k'(T_k^C, t') = \int_0^{t'} r_k(T_k^C, t)\,dt \ ,$$

for a given T_k^C. …The consumed quantity of commodity k increases from zero to x_k' at time t';

Consumption-duration = expected duration of a consumption activity in a day;

Consumption-duration curve = the curve $x_k(T_k^C)$ of a given expectational (intertemporal) plan;

Consumption-rate = R_k, the average consumption over the consumption-duration;

Leisure/rest-duration = balance of 24-hours after total of labor and consumption activities.

Work-duration = expected duration of work activity in a day.

SYMBOL DEFINITIONS

d = day

f_i = function which combines commodities and '*productive*' *consumption* durations to produce basic commodities in Becker's theory.

i = long-run interest rate for periodic equilibrium;

$\mathbf{I_n}$ = unit row vector;

mp^{L1} = marginal productivity of indirect (capital-producing) labor;

mp^{L2} = marginal productivity of direct (output-producing) labor;

$mp^{cap\text{-}L2}$ = marginal productivity of capital;

$\mathbf{p}$ = row vector of market prices;

P = company survival probability at expected future time t;

r_k = instantaneous (instant) rate-of-consumption of commodity k;

R_k = average rate-of-consumption (*consumption-rate*) of commodity k;

$\mathbf{R}$ = diagonal matrix relating commodity quantities to their consumption durations. ($\mathbf{R}$ is identical to $[\boldsymbol{\kappa}^{\mathbf{C}}]$.)

t = expected time;

T = total time per day (24 hours); f

$\mathbf{T^A}$ = column vector of Becker's *"productive" consumption* durations;

$\mathbf{T^C}$ = column vector of commodity consumption-durations;

T_k^C = consumption-duration of commodity k;

$[\mathbf{T^C}]$ = diagonal matrix of consumptive durations;

$[\mathbf{T^C}]^{-1}$ = inverse of $[\mathbf{T}]^{\mathbf{C}}$;

$\mathbf{T_i}$ = partial '*productive*' *consumption* vector residing in $\mathbf{T^A}$-space. ($\Sigma\mathbf{T_i} = \mathbf{T^A}$);

$T^{L/R}$ = leisure/rest-duration;

T^W = work-duration provided to the market;

U = total expected utility;

$\underline{U}^A = \underline{U}^A(\mathbf{T^A})$ = utility function accounting for Becker's '*productive*' *consumption*;

$\underline{U}^W = \underline{U}^W(T^W)$ = utility function for the production (earning) of commodities $\mathbf{x}$;

$\bar{U}^x = \bar{U}^x(\mathbf{x})$ = commodity utility function;

V = interest income;

$\mathbf{x}$ = column vector of purchased and consumed commodities;

$\mathbf{x_i}$ = partial commodity-vector residing in $\mathbf{x}$-space. ($\sum\mathbf{x_i} = \mathbf{x}$: in Becker's theory);

$x_k = R_k\, T_k^C$ = purchased and consumed quantity of commodity k;

Z_i = Becker's "basic commodity." ($Z_i = f_i(\mathbf{x_i}, \mathbf{T_i})$, where $\mathbf{x_i}$ and $\mathbf{T_i}$ conform to $\sum\mathbf{x_i} = \mathbf{x}$ and $\sum\mathbf{T_i} = \mathbf{T^A}$.)

β = generic rate of discounting;

β_I = rate of investment risk (invariant expected chance per day of a sudden company-failure);

β_λ = rate of autonomic discounting;

$\varepsilon_{d\text{-}\beta} = \beta\, d$: for $\beta\, d$ <<1, the change of intertemporal discounting across the single day is negligible;

$\varepsilon_{\beta\text{-}\Lambda} = \Lambda/\beta$: for Λ/β <<1, intertemporal long-run growth or decline is negligible compared to discounting (a necessary condition for expected periodic-equilibrium);

$[\boldsymbol{\kappa}^{C}] = [[\mathbf{T}^{C}]^{-1}\mathbf{x}]$ = diagonal matrix relating commodity quantities to their consumption durations. ($[\boldsymbol{\kappa}^{C}]$ is identical to **R**.)

λ = autonomic discount coefficient.

Λ = expected rate of growth or decline of the company and macro-economy;

ϖ = wage rate;

$[\pm]$ = +1 when the micro-system (company plus manager) is expanding and = −1 when contracting.

$\langle\text{---}\rangle_{\text{Day-x}}$: All bracketed terms have day-x values.

REFERENCES

Becker, G. S. (1962). Irrational behavior and economic theory. *The Journal of Political Economy*, Vol. LXX, No. 1.

Becker, G. S. (1965). A theory of the allocation of time. *The Economic Journal*, Vol. 75, No. 299.

Blaug, M. (1968). *Economic theory in retrospect*. Cambridge: Cambridge University Press. 248-9.

Chamberlain, T. E. (1997). On the psychological basis of economic behavior. Paper presented at the *72nd annual conference of the Western Economic Association International*, Seattle, Washington.

Chamberlain, T. E. (1998a). Mathematical formulation of the expectational plan. Paper presented at the *79th annual meeting of the Pacific Division, American Association for the Advancement of Science*, Logan, Utah.

Chamberlain, T. E. (1998b). On the psychological basis of economics and social psychology. Paper presented at the *38th Congress of the European Regional Science Association*, Vienna, Austria. (Available on the Social Science Research Network: http://ssrn.com/abstract=597341)

Chamberlain, T. E. (1999a). Effect of expectational planning on commodity prices and interest rates. Paper presented at the *47th Bi-Annual Conference of the International Atlantic Economic Society*, Vienna, Austria.

Chamberlain, T. E. (1999b). Small perturbation approach to a transient inter-regional economy accounting for wages, prices, and transaction costs. Paper presented at the *39th Congress of the European Regional Science Association*, Dublin, Ireland.

Chamberlain, T. E. (2000). On the role of subjective uncertainty in the business cycle. Paper presented at the *4th Pacific Rim Allied Organizations Conference*, Sydney, Australia. (Later given at the *40th Congress of the European Regional Science Association*, Vienna, Austria [2000].)

Chamberlain, T. E. (2003/4). Does uneven expected risk promote poverty and instability? Paper presented at the *5th Pacific Rim Allied Organizations Conference*, Taipei, Taiwan. (Later given at IAES/Lisbon [2004], SASE/Washington [2004], and SABE/Philadelphia [2004].) (Available at www.chamberlain-west.com)

Chamberlain, T. E. (2003b). Mortality risk as a basis for time preference. Paper presented at the *55th International Atlantic Economic Conference,* Vienna, Austria.

Chamberlain, T. E. (2003c). Instant utility approach to the social sciences. Paper presented at the *annual meeting of the Society for the Advancement of Behavioral Economics*, Lake Tahoe, CA. (Available at www.chamberlain-west.com)

Chamberlain, T. E. (2006/7a). Relationship of economic stability to social and economic rights. Paper presented at the *61st International Atlantic Economic Conference*, Berlin. (Later given at the *47th Congress of the European Regional Science Association*, Paris, France [2007].) (Available at www.chamberlain-west.com)

Chamberlain, T. E. (2006/7b). Socialism versus Capitalism—economic stability as a unifying goal. Paper presented at the *62nd International Atlantic Economic Conference*, Philadelphia. (Later given at the *7th Pacific Rim Allied Organizations Conference*, Beijing, China [2007].) (Available at www.chamberlain-west.com)

Chamberlain, T. E. (2009b). In search of deeper understanding. *Financial Times* (letter), April 27, 2009.

Chamberlain, T. E. (2009c). True mathematical economics is vital. *Financial Times* (letter), December 18, 2009.

Chamberlain, T. E. (2010/12). History of the Gossen equation leading to unification with neoclassical micro-economics theory. Paper presented at the *69th International Atlantic Economic Conference,* Prague. (Later given at the *85th Western Economic Association Conference*, Portland, Oregon [2010] and the *52nd Congress of the European Regional Science Association*, Bratislava, Slovakia [2012].) (Available at www.chamberlain-west.com) (Title and text revised.)

Chamberlain, T. E. (2011/12). Unified micro/macro-economic theory linking productivity to risk and completing input/output substitution. Paper originally presented at the *71st International Atlantic Economic Conference,* Athens [2011]. (Later given at the *87th Western Economic Association Conference*, San Francisco [2012].) (Title and text revised.)

Clark, J. B. (1899, 1931). *The distribution of wealth*. New York: The Macmillan Company.

Damasio, A. (1994). Decartes' error: emotion, reason, and the human brain. New York: G. T. Putnam's Sons.

DeSerpa, A. C. (1971). A theory of the economics of time. *The Economic Journal*. 81, 828-846.

Dolan, R. J. (2002). Emotion, cognition, and behavior. *Science*, Vol. 298, No. 6761, 8 November 2002.

Georgescu-Roegen, N. (1983). Introduction to Gossen's book. (See Gossen Ref.)

Gossen, H. H. (1854, 1983). *The laws of human relations and the rules of human action derived therefrom*. Translated by Rudolph C. Blitz with an introductory essay by Nicholas Georgescu-Roegen. Cambridge: MIT Press.

Ifcher, J. and H. Zarghamee (2011). Happiness and time preference: the effect of positive affect in a random assignment experiment. *American Economic Review*, Vol. 101, Issue 7.

Jevons, W. S. (1871). *The theory of political economy*. New York: Kelley and Millman, Inc. (1957, 5th edition)

Keynes, J. M. (1936) *The General Theory of Employment, Interest and Money*. Macmillan Cambridge University Press, for Royal Economic Society in 1936.

Knoll, M. (2010). The role of behavioral economics and behavioral decision making in Americans' retirement savings decisions. Social Security Bulletin, Vol. 70 No. 4.

Lange, O. (1936). The place of interest in the theory of production. *Review of Economic Studies*, Vol. 3.

Leontief, W. (1966). *Essays in economics—theories and theorizing*. Volume One. White plains, N.Y.: International Arts and Sciences Press, Inc.

Linder, S. B. (1970). *The Harried leisure Class*. New York and London: Columbia University Press.

Marshall, A. (1890, 1948). *Principles of economics*. 8th ed. New York: Macmillan.

Menger, C. (1871, 1950). *Principles of economics: First general part*. Edited by James Dingwall and Bert F. Hoselitz, with an introduction by Frank H. Knight. Glencoe, Ill.: Free Press.

Ramsey, F. P. (1928). A mathematical theory of saving. *The Economic Journal*. 38, 543-559.

Shackle, G. L. S. (1958) Time in Economics. Amsterdam: North-Holland Publishing Company.

Steedman, I. (2001). *Consumption takes time: implications for economic theory*. Routledge, London.

Walras, L. (1874-1877). *Elements of pure economics: or, the theory of social wealth*. Translated by William Jaffe. Homewood, Ill.: Irwin: London: Allen & Unwin (1954).